Basic idea of RG

Thursday, June 24, 2021 5:29 PM

The overall idea of RG

[Ref: ch 4 Kardar vol 2
ch 9. Goldenfeld]

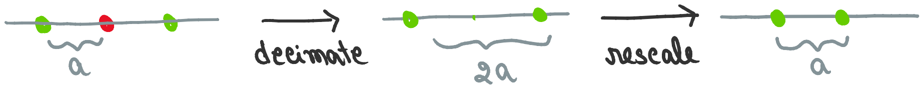
The RG work is based on correct intuition of Kadanoff block-spin coarse graining. At the beggining of 70's K.G. Wilson introduced RG, and showed how to calculate flow of coupling constants, how to explicitly compute critical exponents, and showed how universality emerge.

"The magic of RG is that it helps to identify which big picture quantities are useful to measure and which convoluted microscopic details can be ignored".

Basic steps.

(1) Coarse-graining (decimation): integrate degrees of freedom on scales up to $b.a$ ($a \equiv$ microscopic scale, e.g., lattice unit). This gives us a new effective Hamiltonian.

(2) Rescaling: $\vec{r}_{new} = \frac{\vec{r}}{b}$. This sets the new microscopic scale back to a.



It's immediate from this picture that

Correlation length: $\boxed{\xi_{old} = b\,\xi_{new}}$ This means $\xi_{new} < \xi_{old}$, therefore brings the system away from critical point.

Hamiltonian: $\boxed{H_{old}[\{J\}] \to H_{new}[\{J'\}]}$ From this you could say that a critical fixed point must be repulsive.

Coupling strength: $\boxed{\{J\} \to \{J'\} = R_b[\{J\}]}$

Partition function: $\boxed{Z = \sum_{\sigma} e^{-H_{old}[\sigma]} = \sum_{\hat{\sigma}} e^{-H_{new}[\hat{\sigma}]}}$ remains preserved.

Free energy density: $\boxed{f[\{J'\}] = b^d f[\{J\}] + f_0[\{J, J'\}]}$

Remark: We shall consider analytic transformations R_b, however complicated function it might be.

The operator $R_b[J]$ form a semi-group, because

$$R_{b_1 b_2}[J] = R_{b_1}[R_{b_2}[J]] \quad \text{but } R^{-1} \text{ may not exist.}$$

(many $\sigma \longrightarrow$ one $\hat{\sigma}$)

To be correct it should be called "Renormalization semi group".

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semi-group

① $b_1 \odot b_2 := R_{b_1}\left[R_{b_2}[J]\right] = R_{b_1 b_2}[J] \rightarrow b_1 b_2$ (closure)

② $b_1 \odot b_2 \odot b_3$ is associative.

③ $1 \odot b = b$ because $R_1[J] = J$

④ Inverse does not exist

Remark: Many different decimation proceedures possible. Also can be done in real space or in momentum space. There are three important constraints a decimation must follow

① It leaves partition function invariant

$$\mathcal{Z} = \sum_{\sigma} e^{-H[\sigma]} = \sum_{\hat{\sigma}} e^{-\hat{H}[\hat{\sigma}]}$$

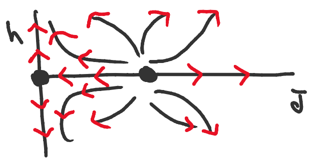
(2) The new Hamiltonian must be real valued.

(3) $\hat{H}[\hat{\sigma}]$ must preserve the symmetry of old $H[\sigma]$.

Remark: The transformation $R_b[J]$ is analytic. Any short scale changes can not lead to singularities. The way singularities arise is by an infinte recursion of this proceedure which leads to stable or unstable fixed points.

Fixed points and linear stability analysis

There could be stable fixed points and unstable fixed points. It could be stable in one direction and unstable in another direction.



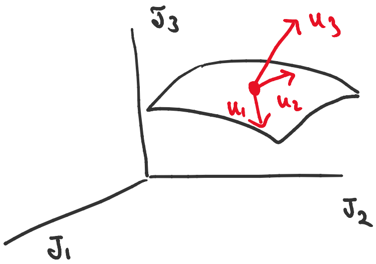
The relation $\xi[J^*] = \frac{1}{b}\xi[J^*]$ at a fixed point J^*

shows that $\xi[J^*]$ is zero or infinite.

Fixed point with $\xi[J^*] = 0$ corresponds to a phase [completely ordered or disordered phase]

A fixed point with $\xi[J^*] \to \infty$ corresponds to a critical point.

Near a critical fixed point $\{J^*\}$:



RG flow:
$$J_i' = \sum_j R_b^{(i)}[\{J\}]\, J_j$$

$$\Rightarrow \quad J_i' - J_i^* = \boxed{\delta J_i' = \sum_j \left.\frac{\partial R_b^{(i)}}{\partial J_j}\right|_{\{J^*\}} \delta J_j} \qquad \text{with} \quad \delta J_j = J_j - J_j^*$$

$$T_{ij}(b)$$

Corresponding eigen vectors

$$T(b) \cdot u_K = \lambda_K(b)\, u_K$$

K is eigenmode index.

Properties: (1) The property $[T(b), T(b')] = 0 \Rightarrow$ eigenvectors u do not depend on b.

(2) $T(b)\,T(b) \equiv T(b^2) \Rightarrow \lambda(b)^2 = \lambda(b^2)$.

In addition, $\lambda(1) = 1$.

$$\Rightarrow \boxed{\lambda(b) = b^y}$$

③ The vector directions u_K are called scaling directions and y_K are called corresponding anomalous dimensions.

u_K are also called scaling fields, and the represent the principlee directions along which flows occur near a fixed point.

(4) Around a fixed point, writing in eigenvector basis

$$\vec{J} = \vec{J}^* + \sum_{K=1}^{n} a_K \vec{u}_K$$

and after linearized RG step

$$\vec{J}' = \bar{R}[\vec{J}] = \vec{J}^* + \sum_{K=1}^{n} a_K \cdot b^{y_K} \vec{u}_K$$

(5) The eigenvector $\vec{u}_K$ is

- relevant if $y_K > 0$: $(a_K b^{y_K})^n$ grows.
- irrelevant if $y_K < 0$: decreases.
- marginal if $y_K = 0$: does not change.

(and one needs to consider beyond linear order)

(6) The correlation length

$$\xi[\delta \vec{J}] = b\, \xi[\delta \vec{J}']$$

in $\vec{u}$ basis

$$\boxed{\xi(a_1, a_2, \cdots, a_n) = b\, \xi(a_1 b^{y_1}, a_2 b^{y_2}, \cdots, a_n b^{y_n})}$$

Repeated RG steps make all the irrelevant amplitudes vanish. On the coupling space it brings on the manifold of rest of the couplings.

Then, the leading singular behavior is determined by relevant and marginal couplings.

Let $u_{m+1}, \cdots, u_n$ are irrelevant couplings. (what are relevant/irrelevant directions depend on the fixed point.) Then, correlation length

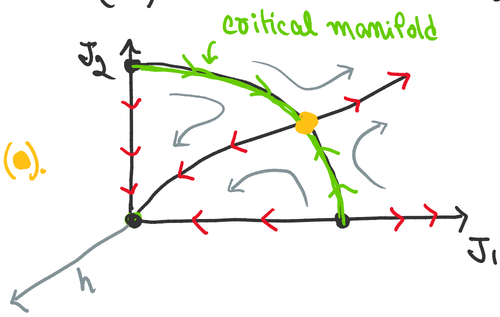
$$
\begin{aligned}
\xi(a_1, \cdots, a_n) &\equiv \xi(a_1, \cdots, a_m) \\
&= b\, \xi(a_1 b^{y_1}, \cdots, a_m b^{y_m}) \\
&= a_1^{-\frac{1}{y_1}} \xi\left(1, a_2 a_1^{-\frac{y_2}{y_1}}, \cdots, a_m a_1^{-\frac{y_m}{y_1}}\right)
\end{aligned}
$$

$b = \frac{1}{a_1^{1/y_1}}$

$$
\Rightarrow \boxed{\xi(a_1, \cdots, a_m) = a_1^{-\frac{1}{y_1}} \Psi_\xi\left(\frac{a_2}{a_1^{\Delta_1}}, \cdots, \frac{a_m}{a_1^{\Delta_m}}\right)}
$$

with $\Delta_k = \frac{y_k}{y_1}$ gap exponents

(7) An example : 2d Ising model with nearest (J_1) and next nearest neighbor (J_2) interactions.



Any point on the critical manifold has infinite ξ and the critical exponents are given by the critical fixed point (●).

This means $(J_1, 0)$ and $(0, J_2)$ both models have same critical behavior.

(8) A critical fixed point is associated with infinite correlation length.

Space of points around a fixed point which flows into the critical fixed point is called the basin of attraction.

For critical fixed point which is repulsive along relevant directions, the basin of attraction is the surface $\{0, \cdots, 0, a_{m+1}, \cdots, a_n\}$ for which relevant amplitudes are zero. Any point on this surface near the critical fixed point flows into the fixed point. This surface is called critical surface or critical manifold.

In the above example of an Ising model, this is the surface

$\{ h=0,$ green line $\}$

Physically this means, if $h=0, J_1 = J^*$ are set, all J_2 small coupling (near critical point) will be critical, namely, has infinite correlation length [for exact proof see 9.3 of Goldenfeld].

This gives the idea of how universality emerge in RG.

How do we see universality?

Only the relevant (and marginal) directions decide critical properties. Any point on the coupling space $\vec{J}_0$ soon converge (around the fixed point) to the manifold of relevant couplings. Then the critical properties are determined by the flow on this manifold.

This means many Hamiltonians with different irrelevant couplings have the same critical behavior given by the relevant couplings common to them.

Remark: Similar to the correlation length ξ, the non-analytic part of free energy density

$$f_{na}(a_1,\cdots,a_m,\underbrace{a_{m+1}\cdots,a_n}_{\text{Irrelevant}}) = a_1^{\frac{d}{y_1}}\,\Psi_f\left(\frac{a_2}{a_1^{\Delta_2}},\cdots,\frac{a_m}{a_1^{\Delta_m}},\underbrace{\frac{a_{m+1}}{a_1^{\Delta_{m+1}}}\cdots,\frac{a_n}{a_1^{\Delta_n}}}\right)$$

For irrelevant directions $\Delta_k = \frac{y_k}{y_1} < 0 \Rightarrow$ as we go closer to critical point $a_1 \to 0$, their contributions vanish.

Then very close to the critical point,

$$f_{na} \longrightarrow a_1^{d/y_1}\,\Psi_f\left(\frac{a_2}{a_1^{\Delta_1}},\cdots,\frac{a_m}{a_1^{\Delta_m}},0,0,\cdots\right)$$

This is fine when the free energy density is analytic around vanishing irrelevant coupling strengths. When this is not, the irrelevant couplings are called dangerously irrelevant variables. One such example is the coupling of the ϕ^4 term in Landau theory (we soon see the details).

Remark: Marginal couplings usually gives logarithmic corrections to critical behavior, and they are important at the upper and lower critical dimensions.

Remark: Although the over all idea was present in Kadanoff's theory, it was not clear how to actually implement in practice, how to deal with the large number of interactions allowed by symmetry, etc. There was a period of uncertainty until Wilson how these steps can be implemented (perturbatively) in the Landau-Ginzburg model.

RG in differential form and the β-functions:

It is useful to treat length scale b as a continuous real parameter. This is less ambigous in continuum formulation of critical phenomena, e.g, Landau theory.

$$J_0 \xrightarrow[b]{} J_1 \xrightarrow[b]{} J_2 \cdots J_n \xrightarrow[b]{} J_{n+1}$$

Denoting lengthscale $l = b^n$ we write $dl = b^{n+1} - b^n = (b-1)l$

It is then more convenient to define a time like variable τ such that

$$l = e^{\tau} \quad \text{and} \quad d\tau = (b-1) \quad \text{and} \quad J_n \equiv J(\tau).$$

Then considering $b \simeq 1$ we can treat τ as continuous variable and define

$$\boxed{\frac{dJ(\tau)}{d\tau} = +\beta\big(J(\tau)\big)}$$

The renormalization beta function.

Then, at fixed point J^*, $\beta(J^*) = 0$ and the linearized RG matrix

$$T_{ij} = \frac{\partial R_i^{(b)}}{\partial J_j}\bigg|_{J^*} = \delta_{ij} + d\tau \cdot \frac{\partial}{\partial \tau} \frac{\partial R_i}{\partial J_j}\bigg|_{J^*} + \mathcal{O}(d\tau^2)$$

$$\frac{\partial \beta_i(J^*)}{\partial J_j} \longleftarrow \frac{\partial}{\partial J_j}\frac{\partial J_i}{\partial \tau}\bigg|_{J^*} \longleftarrow \frac{\partial}{\partial J_j}\frac{\partial R_i}{\partial \tau}\bigg|_{J^*}$$

$$e^{\tau} = b^n$$

This gives

$$T_{ij} \simeq \delta_{ij} + d\tau \cdot \frac{\partial \beta_i(J^*)}{\partial J_j}$$

On the other hand, eigenvalues of T_{ij} are $b^y = (1 + d\tau)^y \simeq 1 + d\tau \cdot y + \cdots$

and eigenvectors u (which does not depend on b, hence on $d\tau$).

This means, y are also the eigenvalues of the matrix $\frac{\partial \beta_i}{\partial J_j}\Big|_{J^*}$ with same eigenvector directions.

Crossover: The scaling predicted theoretically using RG applies very close to the critical point. However in practical examples it may not be possible to reach such theoretical limit, or other relevant variables may be present which could change the scaling behavior. Particularly, reach such theoretical limit, or other relevant variables may be present which could change the scaling behavior. Particularly, scaling exponents may vary smoothly. This is broadly termed as crossover.

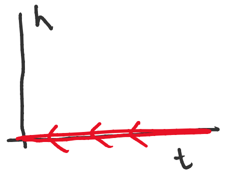
Example 1: Consider the Ising model with renormalized temp t (equivalent to δJ) and magnetic field h. Critical point $(t,h) \equiv (0,0)$.

We showed earlier,

$$\boxed{m(t,h) \sim t^{-\frac{1}{y_1}-\Delta}\, \psi_m\!\left(\frac{h}{t^{\Delta}}\right)} \qquad \text{with } \Delta = \frac{y_2}{y_1}$$

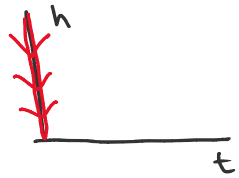
For $h=0$,

$$m(t,0) \sim t^{-\frac{1}{y_1}-\Delta}\, \psi_m(0) \leftarrow \text{constant}$$



gives $m(t,0) \sim t^{\beta}$ with $\boxed{\beta = -\frac{1}{y_1} - \Delta}$

On the otherhand, along $t=0$ line, because $m(0,h)$ is finite



we expect

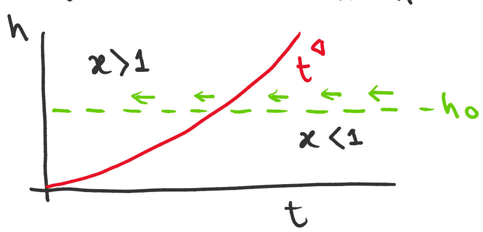
$$m(t,h) \sim t^{-\frac{1}{y_1}-\Delta} \cdot \left(\frac{h}{t^{\Delta}}\right)^{-\frac{\frac{1}{y_1}+\Delta}{\Delta}}$$

$$\sim h^{-1-\frac{1}{\Delta y_1}} \quad \Rightarrow \quad \text{gives } m(0,h) \sim h^{1/\delta} \text{ with } \delta = \frac{\Delta}{\beta}$$

This means

$$\boxed{\begin{array}{lll} \psi_m(x) & \sim x^{-\left(\frac{1}{y_1}+\Delta\right)/\Delta} & \text{for } x \gg 1 \\ & \sim \text{constant} & \text{for } x \ll 1 \end{array}}$$

The behavior of two different limits mean



$$x = \frac{h}{t^{\Delta}}$$

In practical scenario, there is some residual magnetic field, for example earths mag field.

In this scenario, with residual field h_0, when the system is cooled, the critical behavior would start deviating from the ideal theoretical critical behavior predicted for $h=0$ line, i.e., for the exponent β.

The change is not sharp, but rather β would change smoothly: a crossover from one scaling behavior to the other.

A similar applies for other critical exponents. Naturally, this has important consequence in experiment, where it is usually hard to determine critical exponents.

Example 2 A similar such behavior arises when certain relevant fields are not negligible in practice.

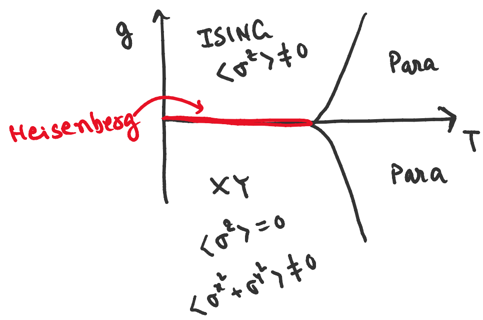
For example, in reality a specific symmetry could be a theoretical idealization, an in practice there may be symmetry breaking terms in the Hamiltonian.

Take a classical Heisenberg model with $O(3)$ symmetry.

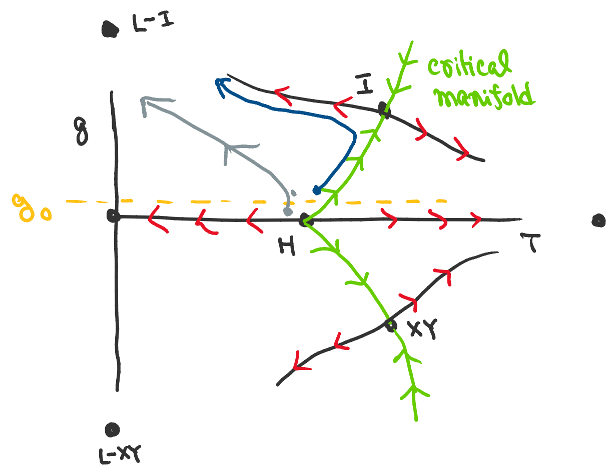
$$\mathcal{H} = -J \sum_{\langle ij \rangle} \bar{\sigma}_i \cdot \bar{\sigma}_j$$ with $\bar{\sigma}$ a three component unit vector.

In practical magnets, there could be anisotropy due to the lattice or anisotropic spin-orbit couplings. For example

$$\mathcal{H} = - \sum_{\langle ij \rangle} \bar{\sigma}_i \cdot \bar{\sigma}_j - g \sum_i (\sigma_i^z)^2$$



$\langle \sigma^x \rangle = \langle \sigma^y \rangle = \langle \sigma^z \rangle = 0$



In the flow diagram

(1) there are 4-stable fixed points, and 3 critical fixed points.

(2) The critical fixed points belong to different universality classes (Heisenberg, XY, and Ising) and their associated critical exponents are different.

(3) A point close to the Heisenberg fixed point (but not on the critical manifold) will flow towards the Ising fixed point (for $g>0$, denoted

(3) A point close to the Heisenberg fixed point (but not on the critical manifold) will flow towards the Ising fixed point (for $g>0$, denoted by blue line) and pass very close to it before moving towards the low temperature fixed point. Then the critical behavior gets determined by the Ising universality class.

(4) On the other hand some other points starting near H-fixed point flows directly towards the low temp phase, thereby critical properties determined by the Heisenberg universality.

(5) Therefore, depending on where one starts in the neighborhood of the heisenberg transition temperature, gets different critical behavior.

(6) This observation in mathematical terms mean

$$\boxed{m(t,g) \sim t^{-\frac{1}{y_1}-\Delta} \, \Psi_m\left(\frac{g}{t^\Delta}\right)} \quad \text{with} \quad \Delta = \frac{y_2}{y_1} > 0.$$

Then, a similar crossover scenario as discussed above for magnetization would arise

$$\text{for} \quad \frac{g}{t^\Delta} \begin{cases} \ll 1 & \text{Heisenberg class} \\ \gg 1 & \text{Ising/XY class} \end{cases}$$

In general, scaling fields which break a symmetry are relevant operators. A high symmetry fixed point is usually (not always) unstable with respect to fixed points of lower symmetry.

Example 3: disorder and Harris criteria.

Real systems have disorder. For example in coupling strength J, lattice defects, etc. How robust are critical properties to the presence of disorder?

In general it is very hard to answer this question.

A.B. Harris (J. Phys. C 7, 1671 (1974)) gave a heuristic criterion for when disorder is not relevant. It is expressed in terms of the critical exponent ν ($\xi \sim 1/t^\nu$) and given as

$$\boxed{\nu d > 2}$$

as long as disorder is local ie disorder spatial correlation decays faster that r^{-d}

When this criteria is not met, disorder becomes a relevant field, and critical properties may flow to a new fixed point.